An introduction to Topological Quantum Computing

Shawn Xingshan Cui
Dept. of Mathematics
Dept. of Physics and Astronomy
Purdue University

Mini course for
Algebraic structures in topology 2026

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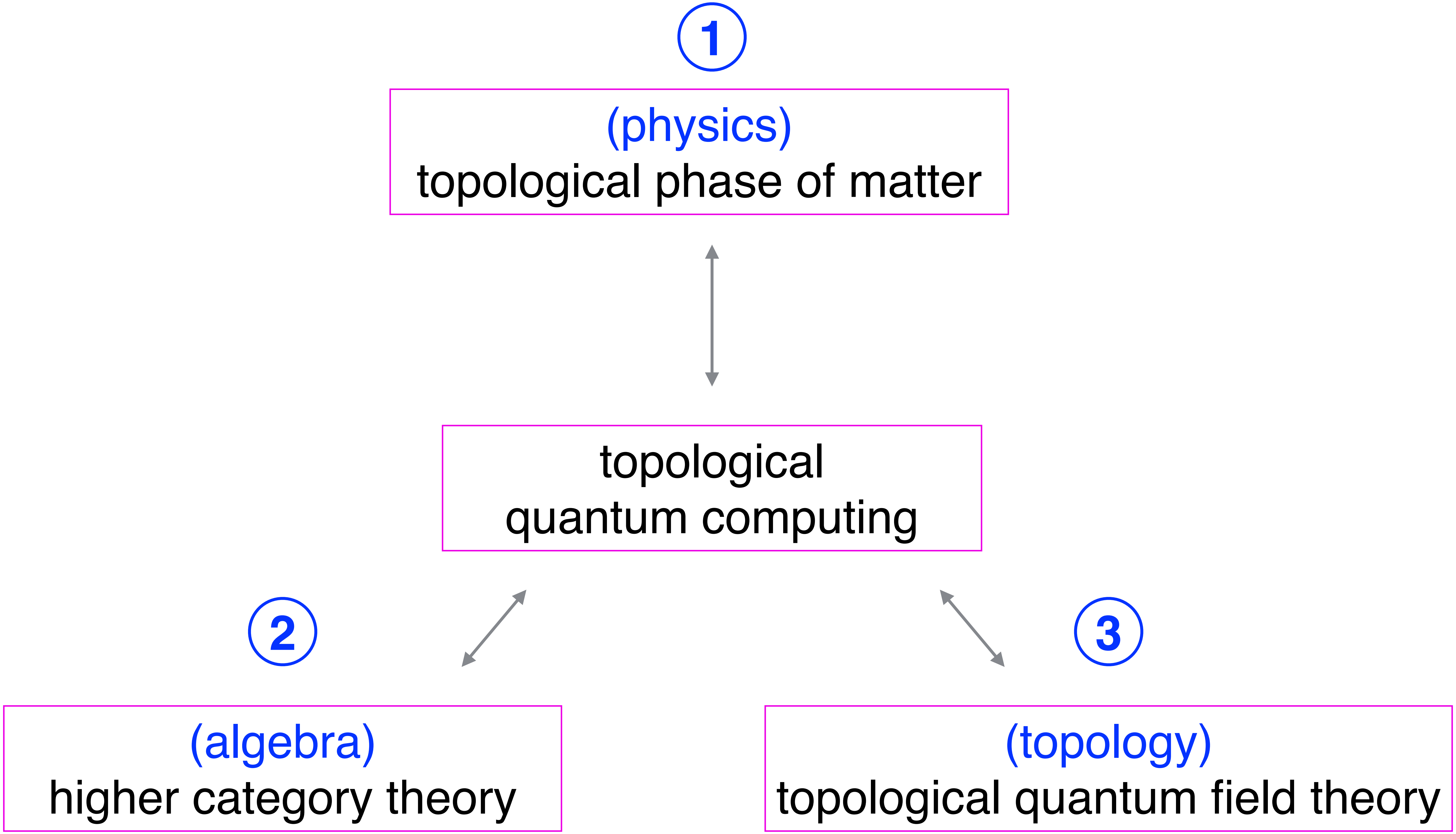

1
(physics)
topological phase of matter
topological
quantum computing
2
(algebra)
higher category theory
3
(topology)
topological quantum field theory

Part III: Outline

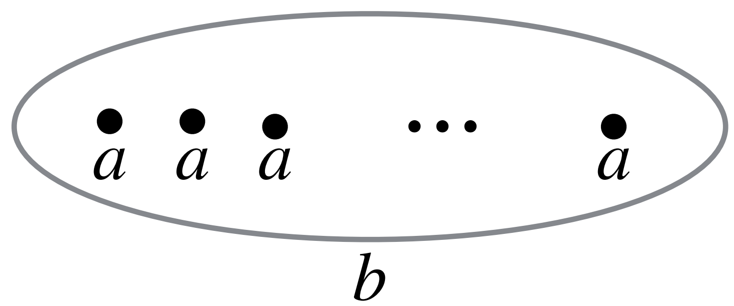
- TQC Setup
- Universality of anyons
- Quantum invariants from anyons

TQC Setup

Fix an anyon system or UMTC $\{L,\ N_{ab}^{c},\ F_{d,ef}^{abc}, R_c^{ba}\}$

★ Qubit spaces and computational bases

$V_c^{a,\cdots,a} \equiv V_b^{a^{\otimes n}}$: space of physical processes of splitting b into n anyons of type a; space of n type-a anyons with total type b

$V_b^{a^{\otimes n}}$ $=$ 

- a must be a non-Abelian anyon

$$\dim V_b^{a^{\otimes n}} \sim (\dim_q a)^n$$

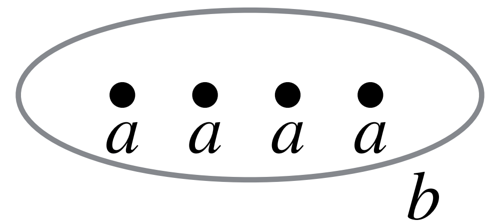
- In general, $V_b^{a^{\otimes n}}$ does not have a tensor product structure. To encode *k* qubits, choose *n* large enough so that

$$(\mathbb{C}^2)^k \hookrightarrow V_b^{a^{\otimes n}}$$

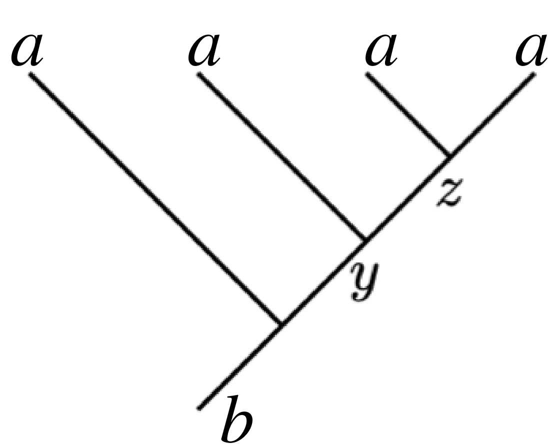
Practically: want a systematic way to choose the embedding

- Choose the `computing' anyon $a$, `total-type' anyon b, and a fixed integer m, such that $V_b^{a^{\otimes m}}$ is the desired qubit/qudit.

e.g. $m = 4$



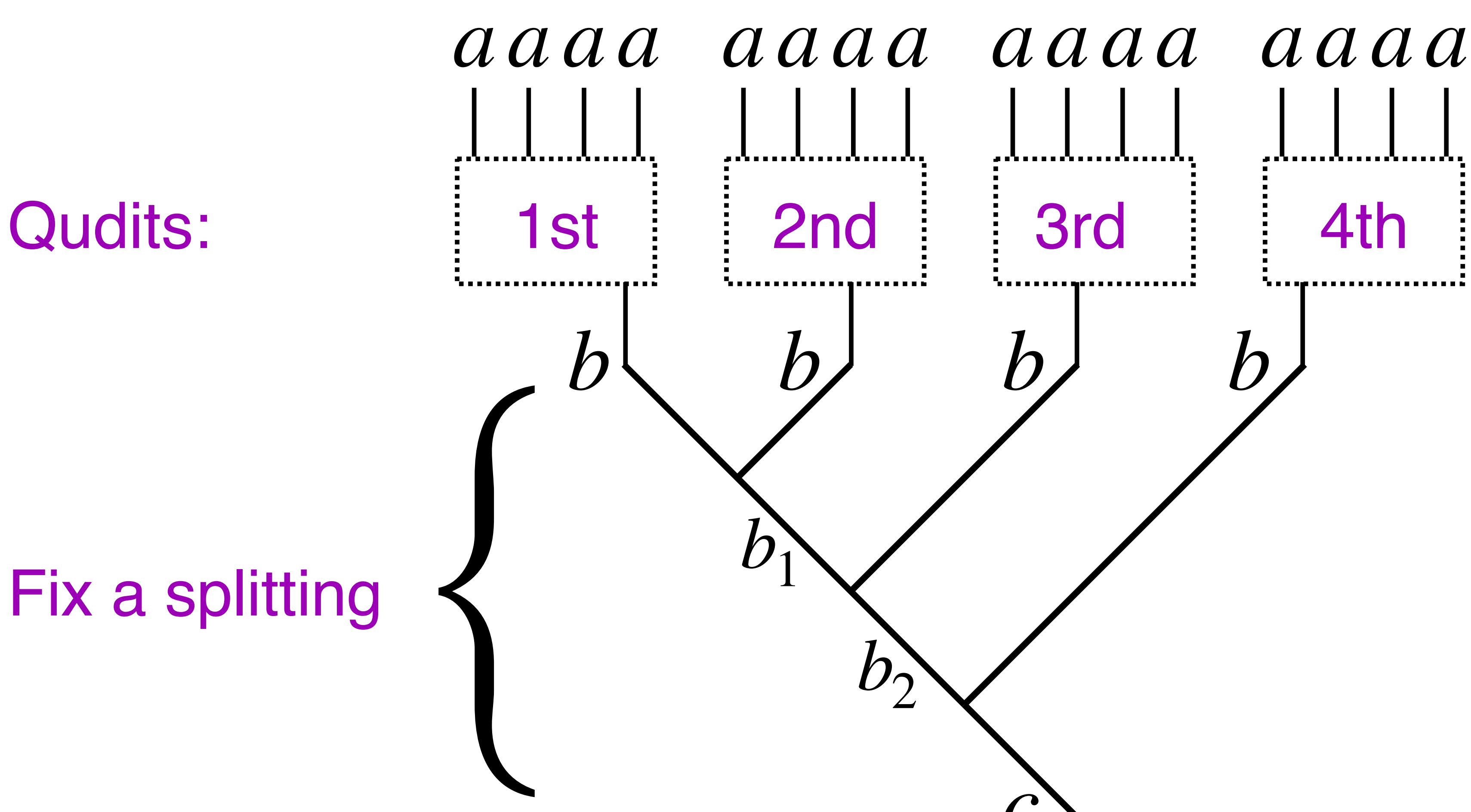
Choose a splitting-tree basis as the computational basis:

$\Bigg\{$ 

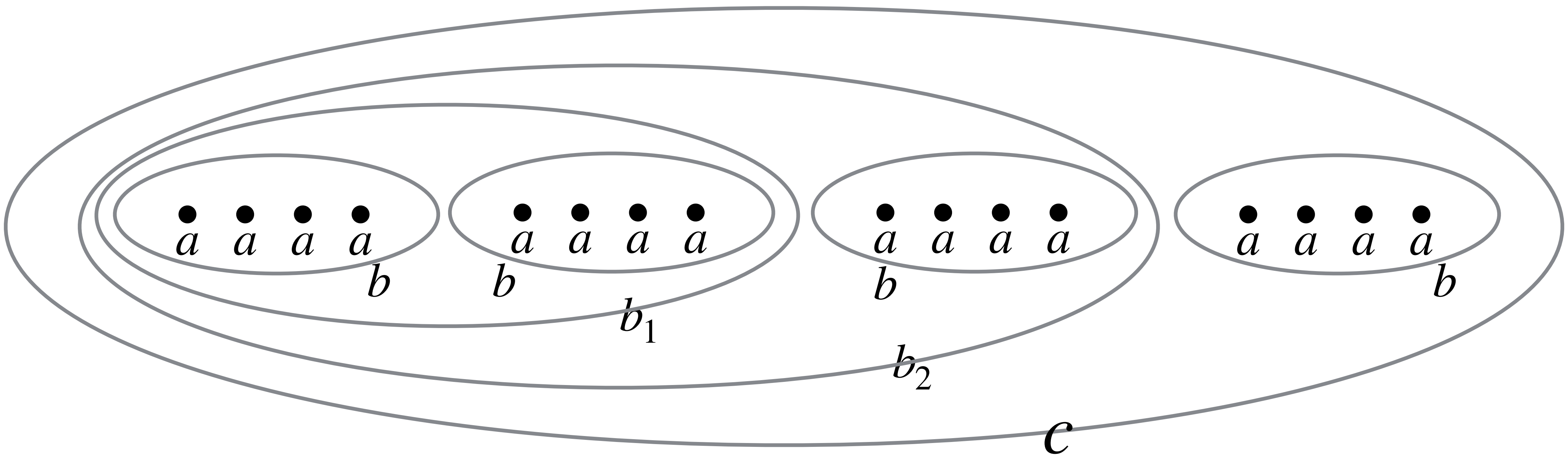
: $(a, a; z), (a, z; y), (a, y; b)$ admissible $\Bigg\}$

- To encode *k* qudits, choose anyon c such that $V_c^{b^{\otimes k}} \neq 0$

$$\underbrace{V_b^{a^{\otimes m}} \otimes \cdots \otimes V_b^{a^{\otimes m}}}_{k} \hookrightarrow V_c^{a^{\otimes mk}}$$



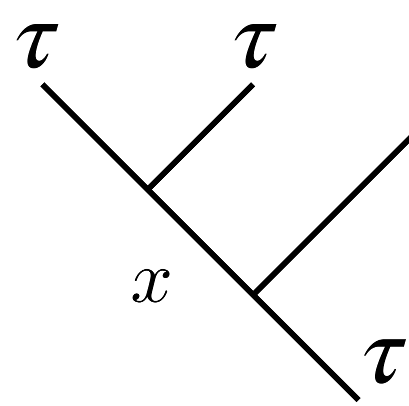
2D picture:



Example: Fib

$$\tau \otimes \tau = \mathbf{1} \oplus \tau$$

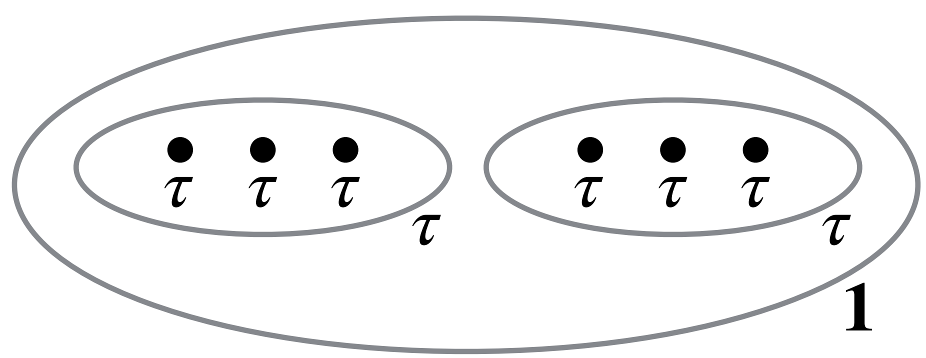
$V_\tau^{\tau\tau\tau}$ has a basis:



$$x = \mathbf{1}, \tau$$

Hence $V_\tau^{\tau\tau\tau}$ is a qubit.

Two qubits: $V_\tau^{\tau^{\otimes 3}} \otimes V_\tau^{\tau^{\otimes 3}} \quad \hookrightarrow \quad V_{\mathbf{1}}^{\tau^{\otimes 6}}$



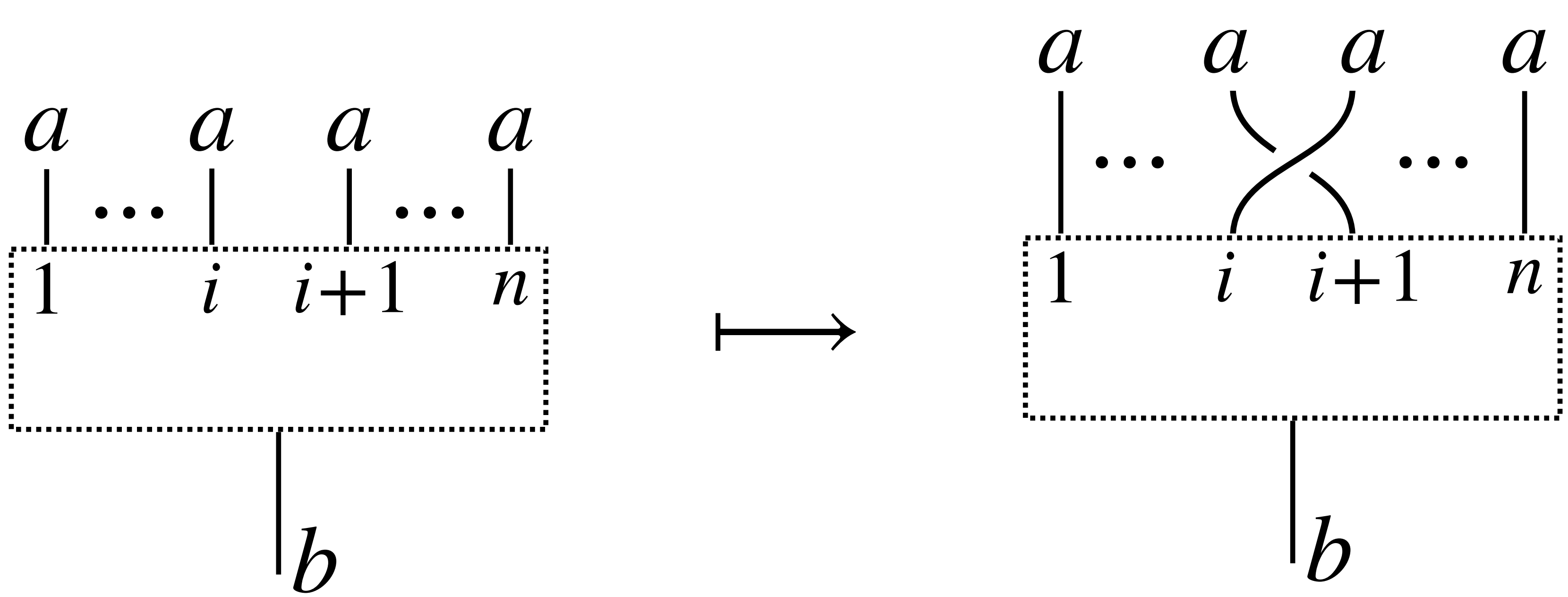
Note $\dim V_\tau^{\tau^{\otimes 6}} = F_5 = 5$

★ Quantum gates from braiding

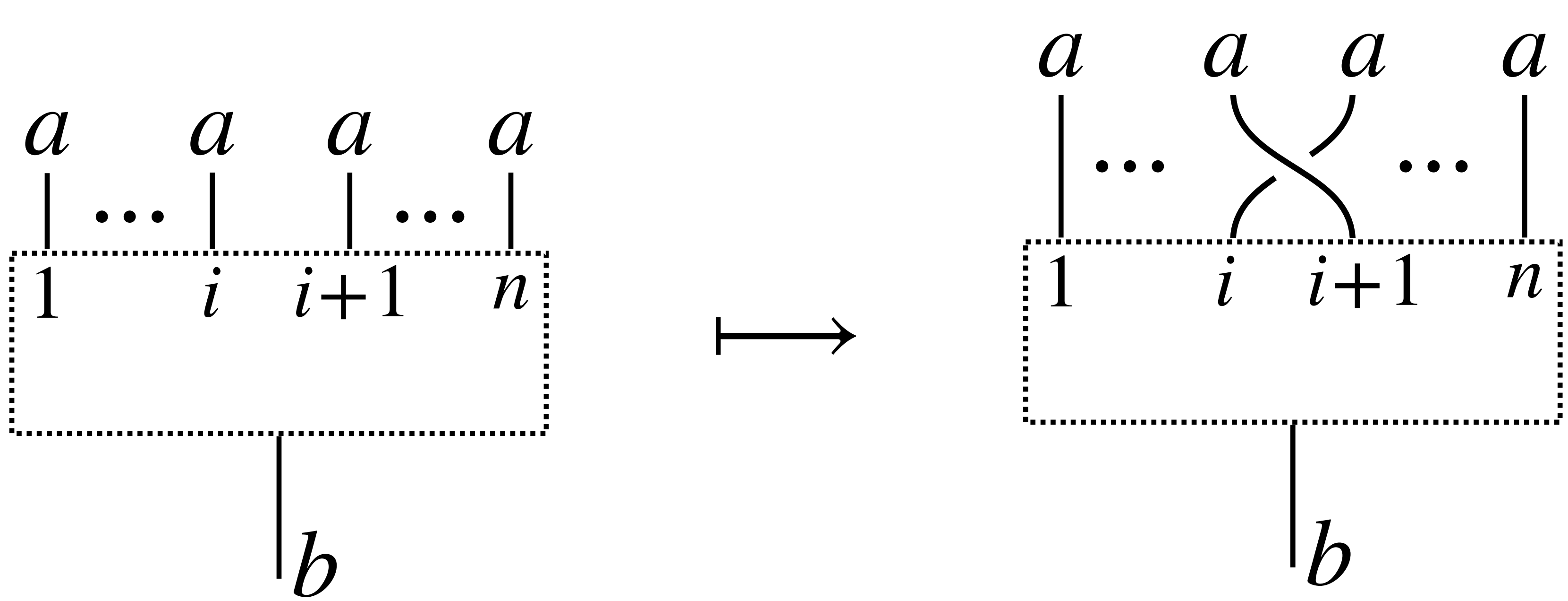
Recall: braiding induces unitary representation of braid groups

$$\rho \equiv \rho_b^{a,n} : B_n \longrightarrow U(\, V_b^{a^{\otimes n}} \,)$$

$\rho(\sigma_i)$:

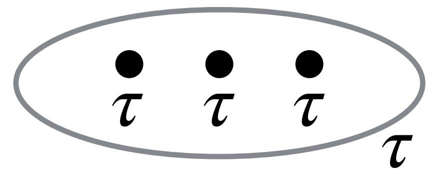


$\rho(\sigma_i^{-1})$:



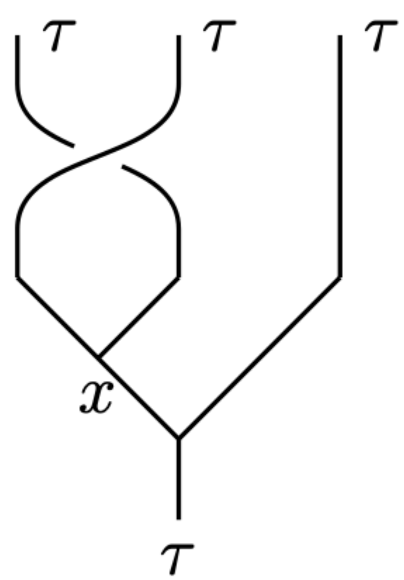
Example: Fib

- 1-qubit gates

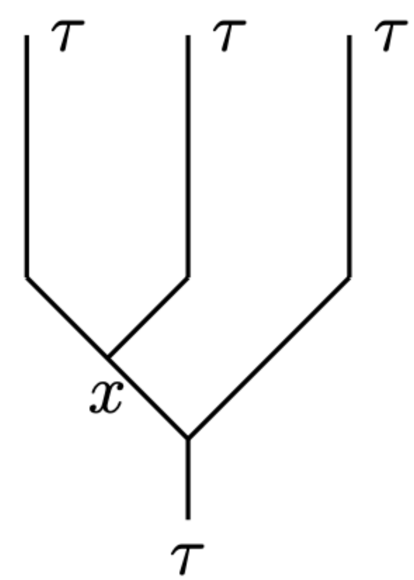


$$\rho : B_3 = \langle \sigma_1, \sigma_2 \rangle \longrightarrow U(\, V_\tau^{\tau^{\otimes 3}} \,)$$

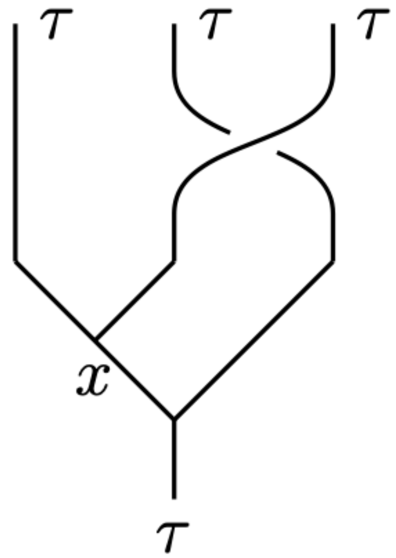
$x = \mathbf{1}, \tau$



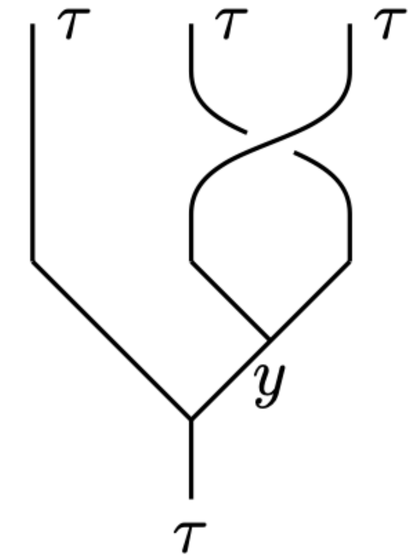
$= R_x^{\tau\tau}$



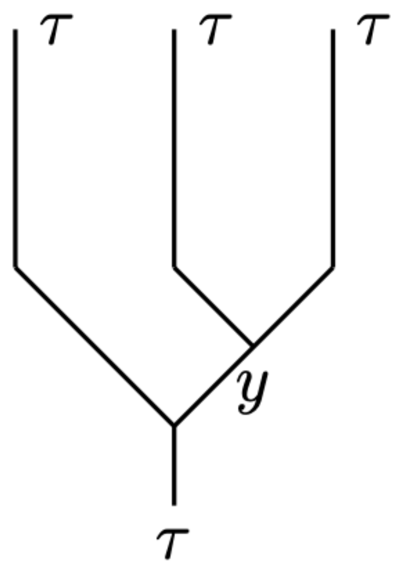
$$\rho(\sigma_1) = \begin{pmatrix} R_{\mathbf{1}}^{\tau\tau} & 0 \\ 0 & R_\tau^{\tau\tau} \end{pmatrix} = \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix}$$



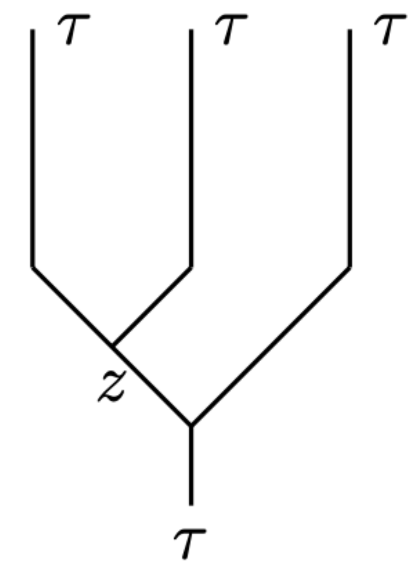
$$= \sum_{y=\mathbf{1},\tau} F^{\tau\tau\tau}_{\tau;yx}$$



$$= \sum_{y=\mathbf{1},\tau} F^{\tau\tau\tau}_{\tau;yx} R^{\tau\tau}_{y}$$



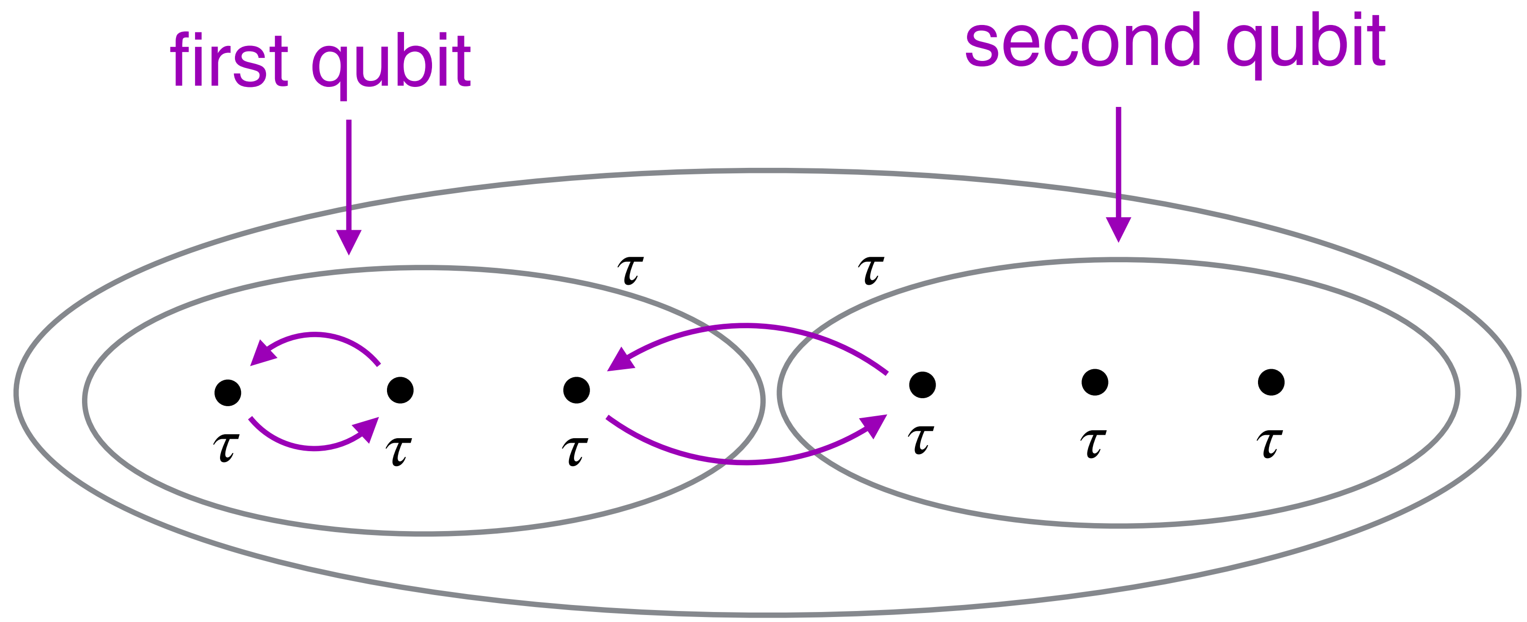
$$= \sum_{y,z=\mathbf{1},\tau} F^{\tau\tau\tau}_{\tau;yx} R^{\tau\tau}_{y} (F^{\tau\tau\tau}_{\tau})^{-1}_{zy}$$



$$\rho(\sigma_2)_{zx} = \sum_{y=\mathbf{1},\tau} F^{\tau\tau\tau}_{\tau;yx}\, R^{\tau\tau}_{y}\, (F^{\tau\tau\tau}_{\tau})^{-1}_{zy}$$

$$\rho(\sigma_2) = (F^{\tau\tau\tau}_{\tau})^{\dagger}\, \rho(\sigma_1)\, F^{\tau\tau\tau}_{\tau}$$

- Multi-qubit gates are computed similarly



2 qubits $\quad V_\tau^{\tau^{\otimes 3}} \otimes V_\tau^{\tau^{\otimes 3}} \subset V_{\mathbf{1}}^{\tau^{\otimes 6}}$

A general braiding acts on $V_{\mathbf{1}}^{\tau^{\otimes 6}}$, not necessarily preserve $V_\tau^{\tau^{\otimes 3}} \otimes V_\tau^{\tau^{\otimes 3}}$

Conjecture

In the above encoding, there is no leakage-free entangling braiding gates.

- A braiding on $V_{\mathbf{1}}^{\tau^{\otimes 6}}$ that preserves $V_\tau^{\tau^{\otimes 3}} \otimes V_\tau^{\tau^{\otimes 3}}$, and
- On $V_\tau^{\tau^{\otimes 3}} \otimes V_\tau^{\tau^{\otimes 3}}$, is not of the form $A \otimes B$ or $\mathrm{SWAP} \circ (A \otimes B)$

Universality of anyons

An anyon a is braiding universal if for some anyon b and for all N large enough, the image of $\mathrm{Im}\,\rho_b^{a,N}$ of the braid group is dense in $U(V_b^{a^{\otimes N}})$.

Theorem (Freedman-Larsen-Wang, 2002)

The τ anyon in Fib is braiding universal.

Theorem (Freedman-Larsen-Wang, 2002)

For the anyon theory SU$(2)_k$ arising from the quantum group $U_q(\mathfrak{sl}_2), \quad q = e^{2\pi i/(k+2)}$, if k is not 0,1,2,4, then the non-Abelian anyons are braiding universal. (Fib $\sim$ SU$(2)_3$)

Property F Conjecture (Naidu-Rowell, 2011)

An anyon has finite braid group image if and only if its quantum dimension squared is an integer. (True for all known cases)

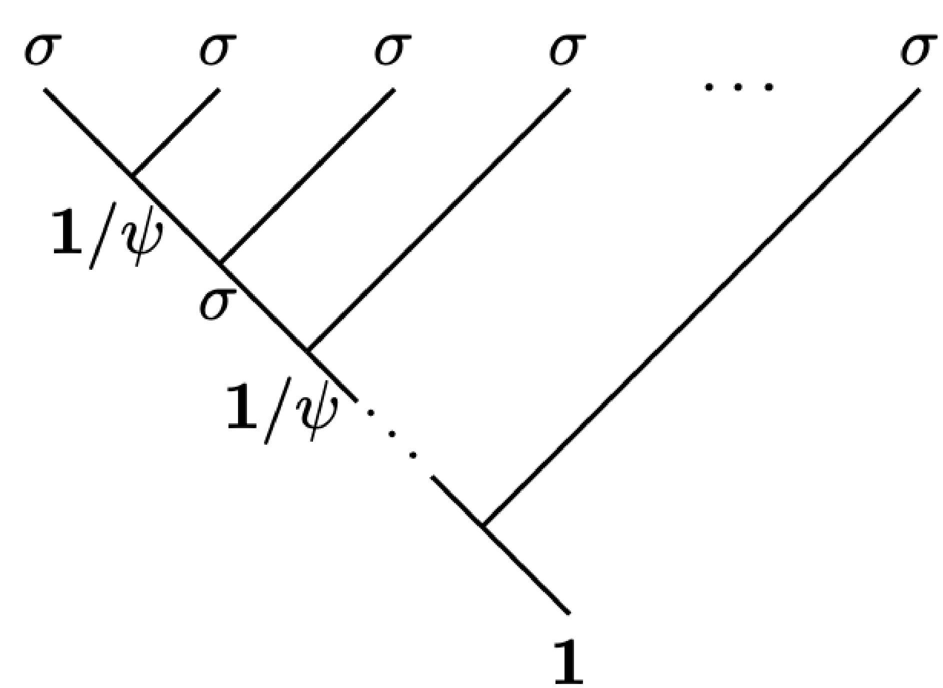
Example: Ising $=$ SU$(2)_2$

- Label set: $L = \{\mathbf{1}, \psi, \sigma\}$; $\bar{a} = a$.
- Fusion rules: $\psi \otimes \psi = \mathbf{1}$, $\psi \otimes \sigma = \sigma$, $\sigma \otimes \sigma = \mathbf{1} \oplus \psi$.
- F-matrices:

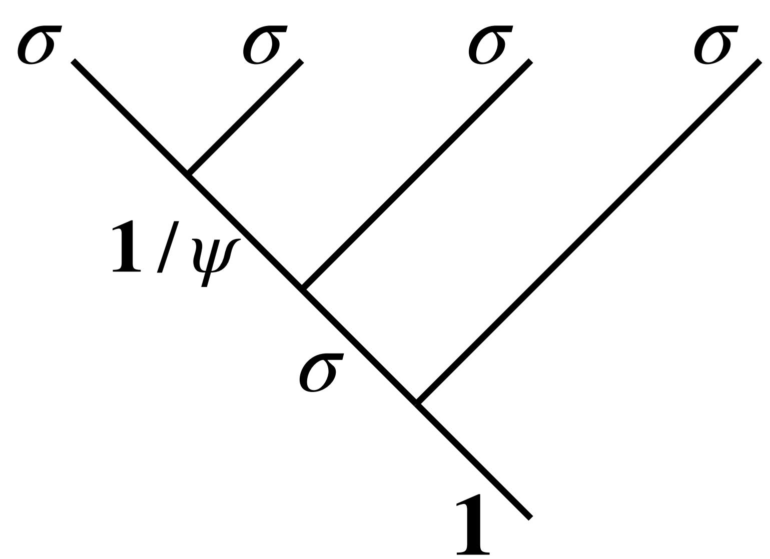
$$F_\sigma^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad F_\sigma^{\psi\sigma\psi} = F_\psi^{\sigma\psi\sigma} = (-1).$$

- R-symbols: $R_\mathbf{1}^{\psi\psi} = -1, \quad R_\sigma^{\psi\sigma} = R_\sigma^{\sigma\psi} = -i, \quad R_\mathbf{1}^{\sigma\sigma} = e^{-\pi i/8}, \quad R_\psi^{\sigma\sigma} = e^{3\pi i/8}.$

$\dim V_{\mathbf{1}}^{\sigma^{\otimes 2n}} = 2^{n-1}$, a rare case!



Qubit $V_{\mathbf{1}}^{\sigma\sigma\sigma\sigma}$:



$$\rho(\sigma_1) = \rho(\sigma_3) = e^{-\frac{\pi i}{8}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

$$\rho(\sigma_2) = e^{-\frac{\pi i}{8}} \begin{pmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{pmatrix}$$

Fact: $\mathrm{Im}\,\rho_{\mathbf{1}}^{\sigma,2n}$ is the Clifford group generated by $\{\mathrm{CNOT}, H, P, X\}$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Theorem (Gottesman–Knill)

Quantum gates in the Clifford group can be simulated efficiently on a classical computer.

A quantum computer built from the Ising anyons have no quantum advantage.

Example: SU$(2)_4$

Anyon types $L = \{0,1,2,3,4\}$

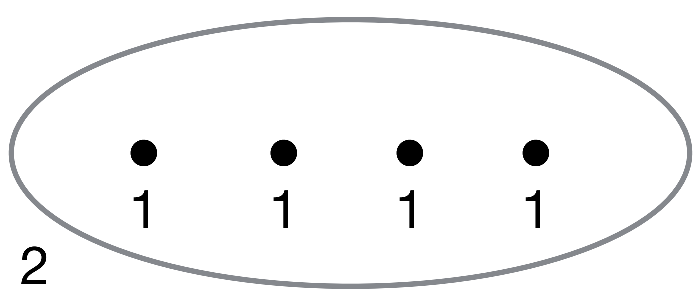
vacuum

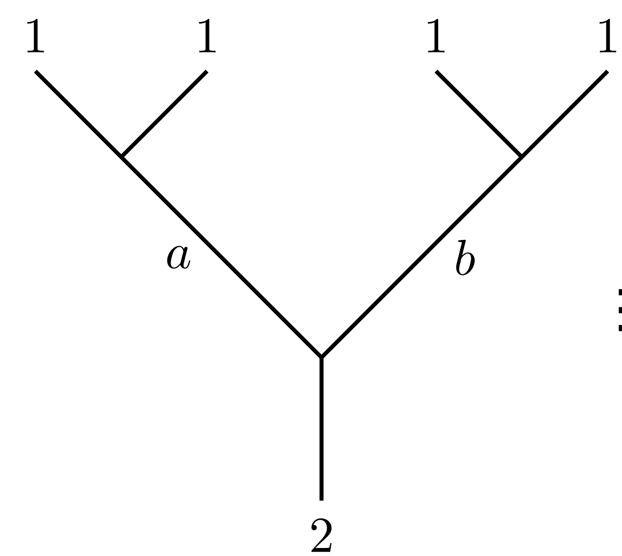
Partial fusion rules:

$$1 \otimes 1 = 0 \oplus 2$$

$$2 \otimes 2 = 0 \oplus 2 \oplus 4$$

- qutrit model



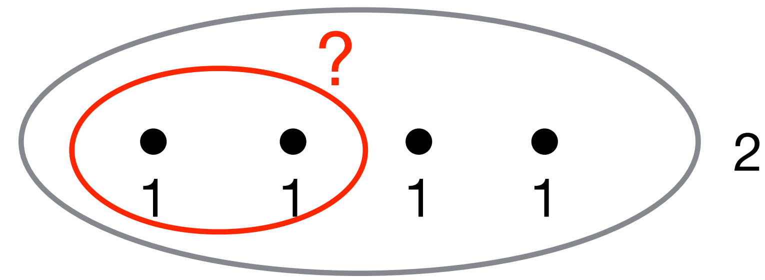


$\equiv |a, b\rangle$

$$(a, b) = (0,2),\ (2,0),\ (2,2)$$

The braid group has finite image — not braiding universal

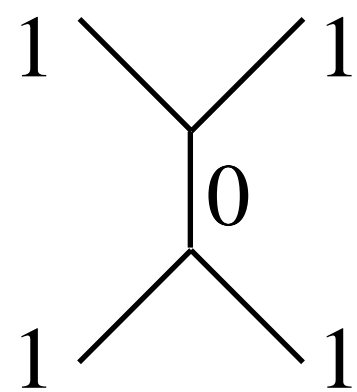
- Measurement of total type



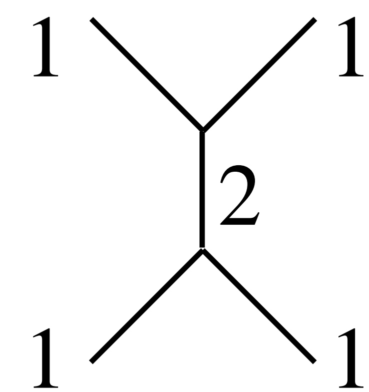
$$1 \otimes 1 = 0 \oplus 2$$

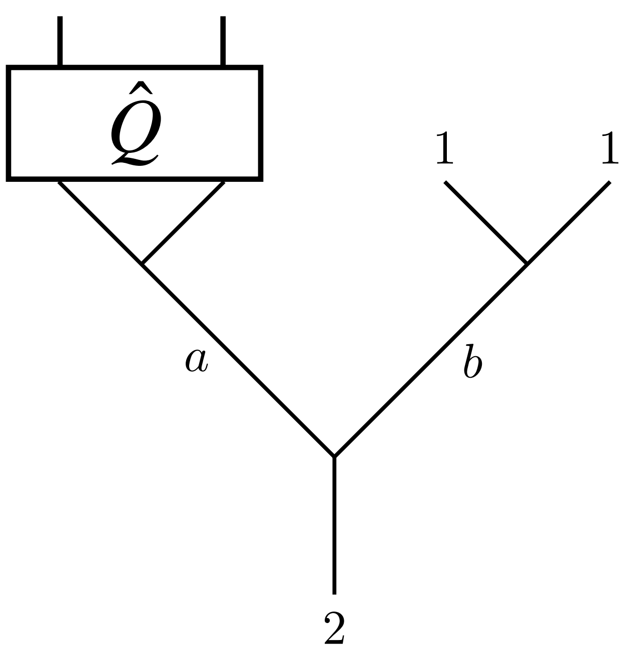
For some $\lambda_0 \neq \lambda_2 \in \mathbb{R}$

$\hat{Q} \;:=\; \lambda_0$

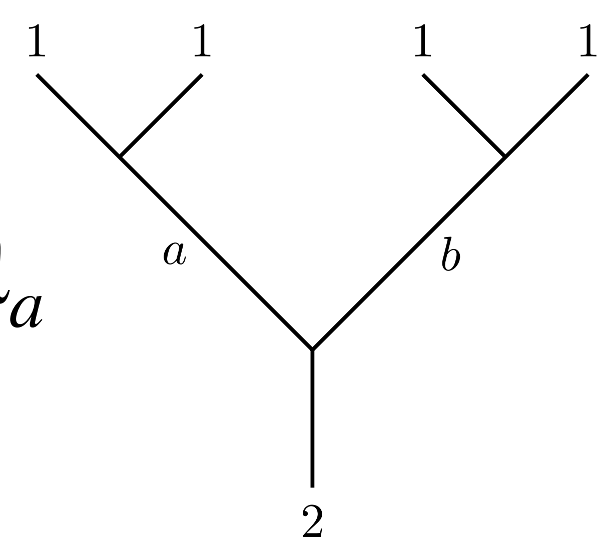


$+ \quad \lambda_2$

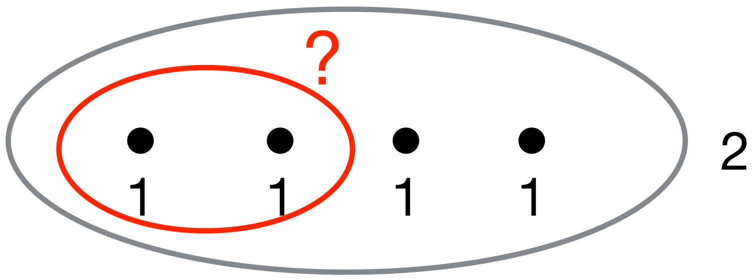




$= \quad \lambda_a$



$(a, b) = (0,2),\ (2,0),\ (2,2)$



Effect of measuring the total type of the two type-1 anyons, i.e. $\hat{Q}$:

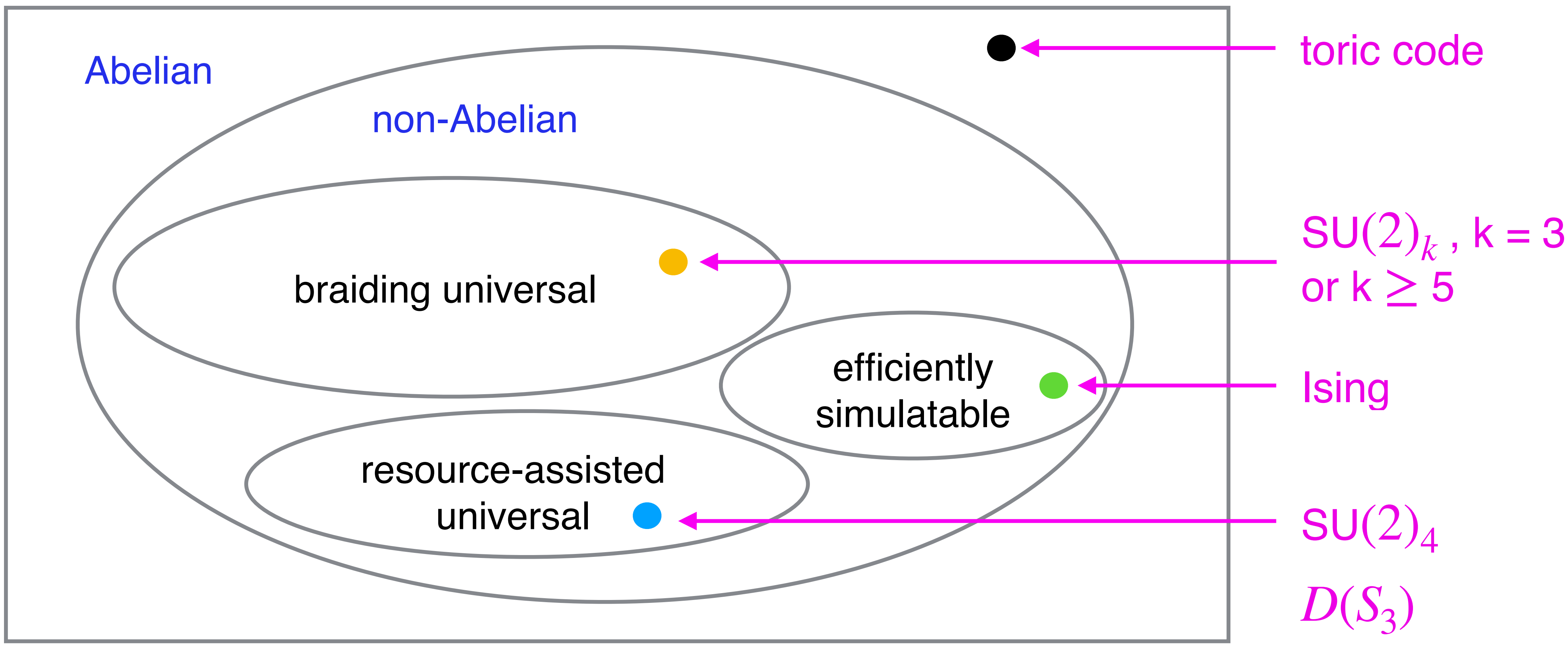
$$\alpha|0,2\rangle + \beta|2,0\rangle + \gamma|2,2\rangle \quad \begin{cases} \nearrow \ |0,2\rangle & \text{Prob} = |\alpha|^2 \\ \searrow \ \dfrac{\beta|2,0\rangle + \gamma|2,2\rangle}{|\beta|^2 + |\gamma|^2} & \text{Prob} = |\beta|^2 + |\gamma|^2 \end{cases}$$

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$$

Theorem **(Cui - Wang, '14)**

In $\mathrm{SU}(2)_4$ with the above encoding, braiding supplemented with the measurement of the total type of two anyons, is universal —

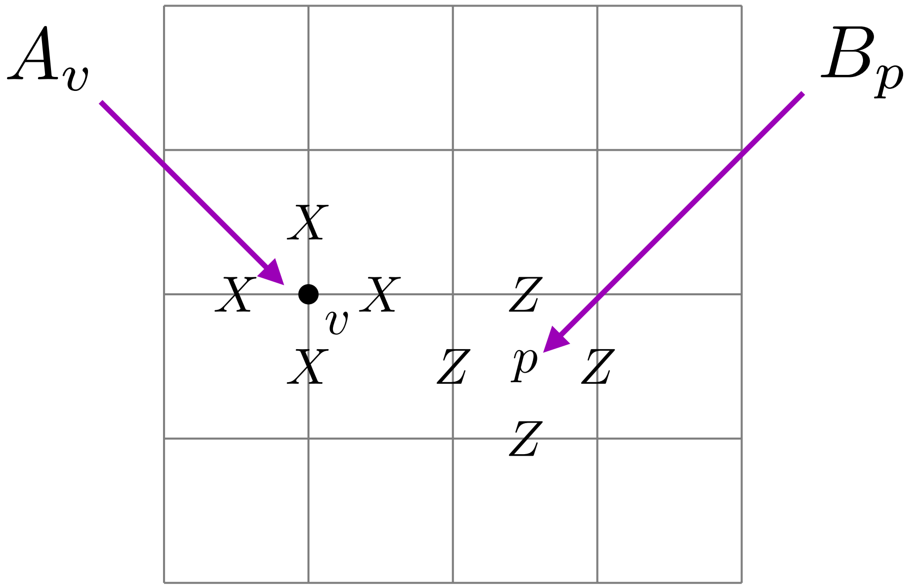
Resource-Assisted Universality



Problem

How to characterize each class of anyons?

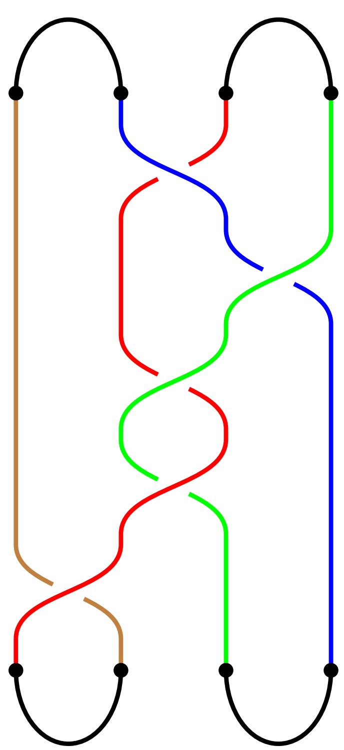
Topological quantum field theory (TQFT)



Put the system on a general surface

$$\Sigma \longmapsto V(\Sigma)$$

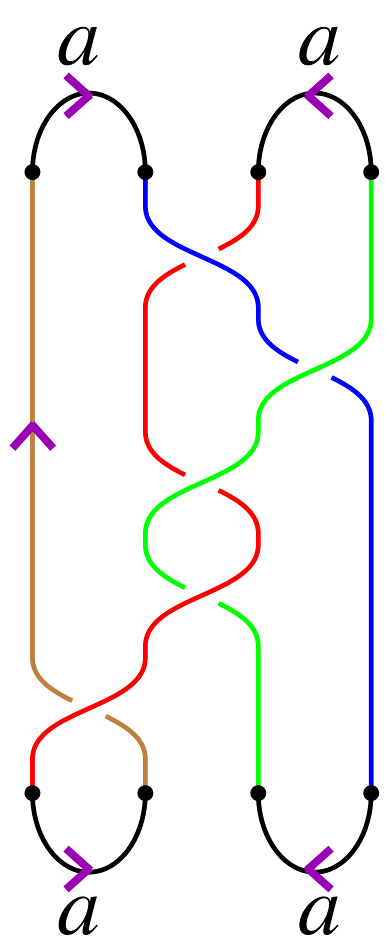
vector space of states



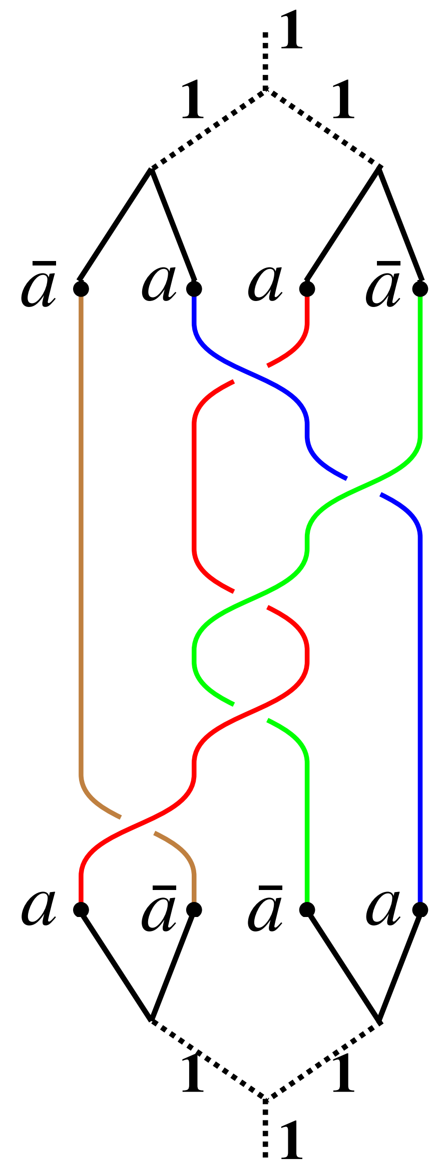
$$(M;K) \longmapsto Z(M;K)$$

Knot in a 3-manifold — Knot invariant

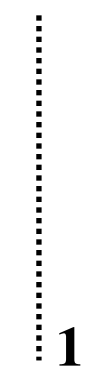
For an oriented knot/link K labelled by anyon a



$=$ $(\dim_q a)^2$



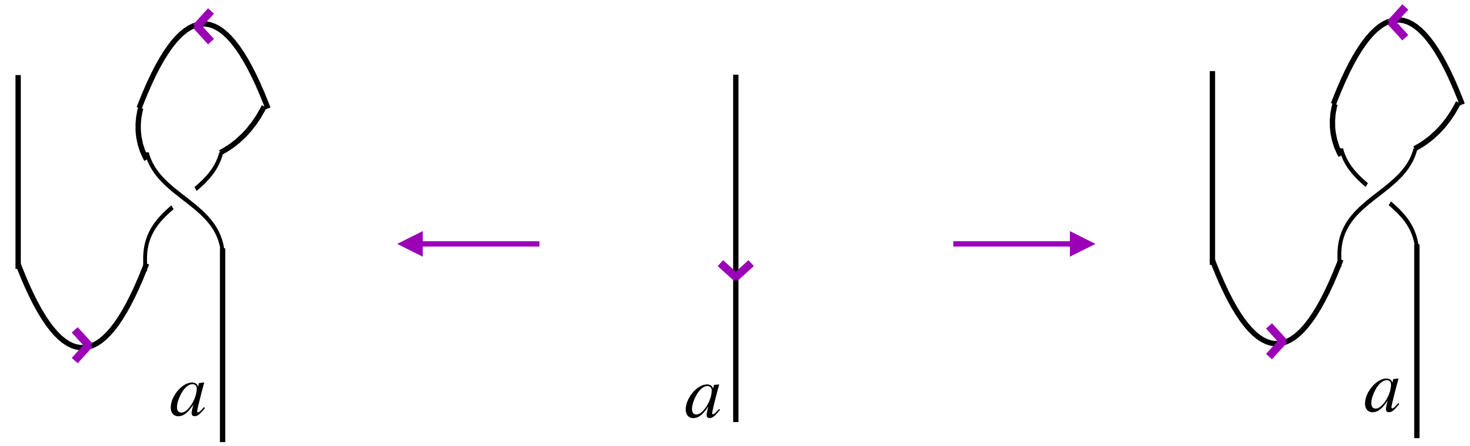
$=: \langle K; a \rangle$



- Can be computed using *F*- and *R*-symbols.
- Invariant under Reidemeister II and III moves.

$\langle K; a \rangle$ is an invariant of framed oriented links — Reshetikhin-Turaev invariant

- Reidemeister I move:



$$\theta_a \langle K;a\rangle \longleftarrow \langle K;a\rangle \longrightarrow \overline{\theta_a}\,\langle K;a\rangle$$

topological twist

Let $lk(K)$ be the linking number of K,

$$J(K;a) := \theta_a^{-lk(K)}\,\langle K;a\rangle$$

is an invariant of un-framed oriented links

Fact: for the Fibonacci anyon, $J(K;\tau)$ is (a version of) of the Jones polynomial of K evaluated at 5-th root of 1.

$\mathsf{Vec}_{\mathbb{C}}$: category of vector spaces

objects: vector spaces, $V_1, V_2, \cdots$

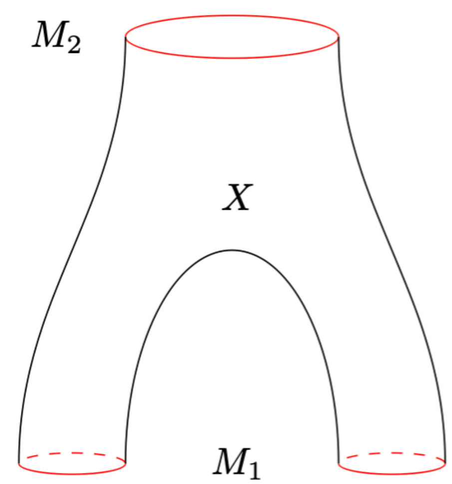
morphisms: linear maps $f : V_1 \to V_2$

$\mathsf{Cob}(d+1)$: category of cobordisms

objects: closed d-dim manifolds, $M_1, M_2, \cdots$

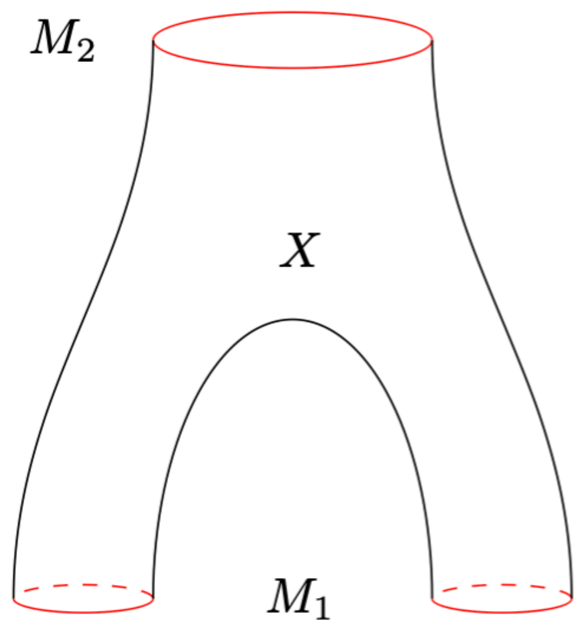
morphisms: $(d+1)$-dim cobordisms $X : M_1 \to M_2$

i.e. $\partial X = M_1 \sqcup M_2$



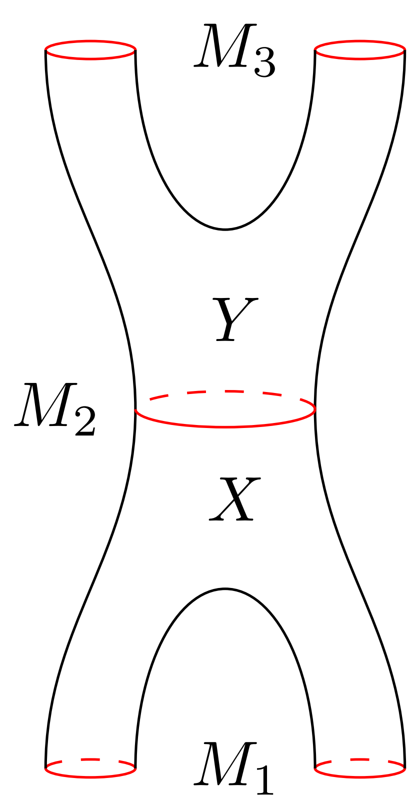
A $(d+1)$-dim topological quantum field theory (TQFT) is a symmetric functor

$$(V, Z) : \mathsf{Cob}(d+1) \ \to \ \mathsf{Vec}_{\mathbb{C}}$$



$$\begin{tikzcd} V(M_2) \\ V(M_1) \arrow[u, "Z(X)"] \end{tikzcd}$$

Subject to (for instance)



$$\begin{tikzcd} V(M_3) \\ V(M_2) \arrow[u, "Z(Y)"] \\ V(M_1) \arrow[u, "Z(X)"] \arrow[uu, bend right, "Z(X \cup_{M_2} Y)"'] \end{tikzcd}$$

- If X is closed, i.e., no boundary,

$$X : \varnothing \to \varnothing,$$

$$Z(X) : \mathbb{C} \to \mathbb{C}, \qquad Z(X) \in \mathbb{C}$$

$Z(\,\cdot\,)$ is an invariant of closed (d+1)-manifolds — quantum invariants

- Given $\phi : M \xrightarrow{\sim} M$, let M_ϕ be the mapping cylinder,

$$M_\phi : M \to M$$

$$Z(M_\phi) : V(M) \to V(M)$$

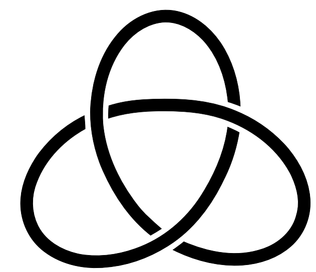
representation of the mapping class group: $\mathrm{MCG}(M) \to \mathrm{GL}(\ V(M)\ )$

$$\phi \quad \mapsto \quad Z(M_\phi)$$

(2+1)-dim TQFTs

Modular tensor category $\mathscr{C}$ $\longrightarrow$ Witten-Reshetikhin-Turaev $Z_{\mathscr{C}}(\,\cdot\,)$

Let *M* be a 3-manifold and *L* be a surgery link for *M*

$$Z_{\mathscr{C}}(M) \;=\; \sum \quad \dim_q(x) \quad \Big\langle$$  $$^{x} \Big\rangle$$

anyon types $x \in \mathscr{C}$ weight Reshetikhin-Turaev invariant

Volume Conjectures (Chen-Yang, '15)

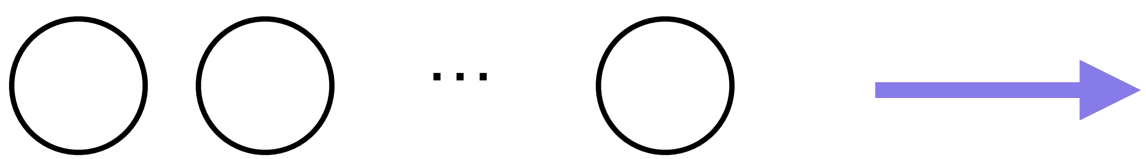
Let $\mathrm{TLJ}(r)$ be the Temperley-Lieb-Jones category with the Kauffman variable $A = e^{\frac{\pi i}{r}}$, *M* be a closed oriented hyperbolic 3-manifold. Then for r odd,

$$\lim_{r\to\infty} \frac{4\pi}{r} \log|Z_{\mathrm{TLJ}(r)}(M)| \;=\; \mathrm{Vol}(M)\,.$$

(3+1)-dimensions

Topological phases of matter:

- Anyons (point-like excitations) can only be fermions/bosons
- More interesting excitations are string-like. Motions of strings lead to representations of the motion groups of links.



N-component unlink

Loop braid group LB_n

$$\langle \sigma_1, \cdots, \sigma_{n-1}, s_1, \cdots s_{n-1} \mid \cdots \rangle$$

Algebraic theories: monoidal/braided 2-categories

Special cases: ribbon fusion categories, group-crossed braided fusion, spherical fusion 2-categories, etc.

(3+1)-TQFTs:

Construct TQFTs to distinguish smooth structures of 4-manifolds.

- Turaev-Viro type based on fusion 2-categories [Douglous-Reutter, ’18]

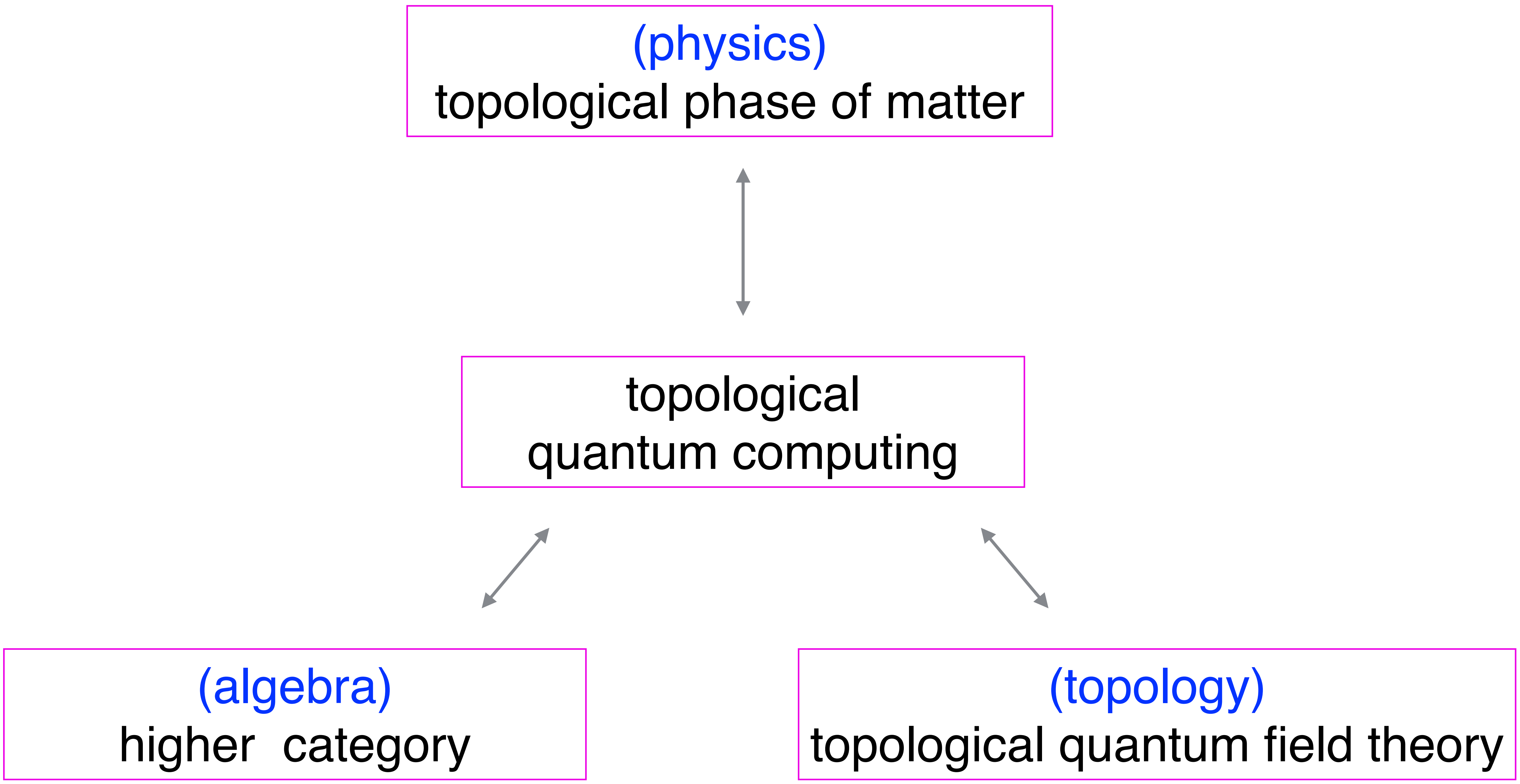
No-go theorem:

Semisimple TQFTs cannot distinguish smooth structures [Reutter, ’20]

Non-semisimple efforts:

- Trisection invariant based on Hopf algebras [Chaidez-Cotler-Cui, ’19, ’23]
- Skein TQFTs from ribbon categories [Costantino-Geer-Haïoun-Patureau-Mirand, ’23]
- Invariants of 4D 2-handlebodies under 2-equivalence from ribbon cats [Beliakova-De Renzi-Faes, ’25; Beliakova-De Renzi, 22’]
- Khovanov homology and skein lasagna modules [Morrison-Walker-Wedrich, ’22; Ren-Willis, '24]

can distinguish smooth structures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